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# On the invertible objects in tensor categories

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**Abstract.** The invertible objects in a tensor category form a subcategory the Grothendieck ring of which is the group ring of an abelian group. This abelian fusion ring acts on the objects of the initial category and one can in principle determine all $6j$-symbols that contain the lable of an invertible object.

**PACS.** 02.20.Uw Quantum groups

## 1 Introduction

Symmetries in physics and the group or algebra objects standing behind them are a very well developed area. On the other hand tensor categories still is considered an exotic subject to be taught in a physicists curriculum. By the Tannaka-Krein duality the group and the category of its representations contain the same information. And in fact it is exactly the category side of this duality which is most natural for physical models. The categorical data enters directly in such models or even can be measured in experiments (say the Clebsch-Gordon rules). Another reason to consider seriously tensor categories is that many generalizations (like passing from a symmetric to a braided category) are more natural in the language of categories than in their algebraic Tannaka duals. In the case of rational conformal field theory models the relevant categories, which are not Tannakian in the narrow sense of the word, were described by Moore and Seiberg seven years before the correct algebraic duals, the correct quantum symmetries (the weak Hopf algebras of Böhm and Szlachanyi) were discovered.

Thus it is worthwhile to try to understand better tensor categories. (Good references for the notions discussed here are the lectures [2] or the textbook [23].) The structure theory describing notions like subgroups, normal subgroups, simple groups, composition series, etc, are central in group theory. One would like to have a similar structure theory in tensor categories. Categories where all objects are invertible are fully characterized by their fusion rules and a third cohomology class (of the abelian group cohomology) characterizing the solution of the pentagon equation. Here we consider the simplest possible case of structure theory when we have a tensor category containing a subcategory of invertible objects.

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## 2 Fusion rings

The definition of a fusion ring $R$ [9,10,6] is an abstraction of the properties of the Grothendieck ring $K_0(\mathtt{C})$ of a rigid braided semisimple monoidal category $\mathtt{C}$ (a tensor category for short). The functor $K_0$ forgets about morphisms, in particular about isomorphisms, *i.e.*, $K_0(x) = K_0(y)$ for isomorphic objects $x \simeq y$. Abusing notation we denote both the object $x$ and the ring element $K_0(x)$ by $x$ and hence $K_0(x \otimes y) = x \cdot y$ and $K_0(x \oplus y) = x + y$. For certain issues it is convenient to pass to an algebra (over the complex numbers) thus a fusion algebra $R$ is a unital associative and commutative algebra with a chosen basis $S$ such that the fusion rules $N_{xy}^z$, $x, y, z \in S$, *i.e.*, the structure constants in this basis, $x \cdot y = \sum_z N_{xy}^z z$, are in $\mathbf{Z}_+$ and their is an involutive automorphism $x \to \bar{x}$ such that $N_{xy}^1 = \delta_{\bar{x},y}$. The set $S$ corresponds to the sectors or the spectrum of $\mathtt{C}$, *i.e.*, the equivalence classes of simple objects or irreps, the monoidal structure in $\mathtt{C}$ is responsible for the structure of unital associative ring, in particular $N_{xy}^z = \dim(\mathrm{Hom}(x \otimes y, z))$, the braiding for the commutativity, while the rigidity translates in the involutive automorphism.

Fusion rings or algebras appear in many occasions (here we consider only the case of finite dimensional ones): the category $\mathtt{C}$ in $K_0(\mathtt{C})$ could be $\mathbf{Rep}$(finite (quantum) group); $\mathbf{Rep}(U_q(g))/Z$ with $q^p = 1$, $g$ a simple Lie algebra, and $Z$ the ideal of zero quantum dimensional modules; also $\mathtt{C}$ could be the Moore-Seiberg category of 2-dimensional rational conformal field theory (2D-RCFT) or the Doplicher-Roberts category of localizable automorphisms of the algebra of observables of a 2D-QFT (quantum field theory) with $S$ labeling the superselection sectors (the generalized charges). The last three are typically non Tannakian categories and in particular the statistical dimensions(=ranks) of the sectors are in general only algebraic integers. Most generally $\mathtt{C}$ is the rep category of a quasitriangular weak Hopf algebra (or quantum groupoid). On many occasions (2D-RCFT, 2D-QFT) one has more structure with C being ribbon(=tortile) and in fact a Turaev modular category with $S$ comprising a representation of the modular group $SL_2(\mathbf{Z})$ with modular $S$- and $T$-matrices. The $S$-matrix plays the role of characters and diagonalizes the fusion rules (Verlinde's famous formula) while $T$-matrix is diagonal with the balancing phases on the diagonal.

Fusion algebras are particular cases of table algebras [1]. For a table algebra the requirements that the structure constants $N_{xy}^z$ are positive integers and $N_{xy}^1 = \delta_{\bar{x},y}$ are relaxed to $N_{xy}^z \in \mathbf{R}_+$ and $N_{xy}^1 \neq 0$ iff $\bar{x} = y$. Table algebras have been extensively studied by Arad, Blau and coworkers. Particular cases of table algebras with generators of dimension 2 or 3 have been classified.

For finite groups it is clear that simple groups have fusion rules algebras which have no notrivial subfusion rule algebras, hence such fusion rule algebras is natural to call simple. More generally if a group $G$ has a normal subgroup $H$ then $K_0(G/H)$ is a subfusion rule algebra of $K_0(G)$. This extends to Hopf algebras [19] and [20]. For table algebras there is a more developed structure theory [3] – in particular one has composition series for table algebras. What is the theory of extensions for fusion rule algebras is an open subject. Since $K_0$ is only half exact one will probably have to use the higher $K$-functors and the long exact sequence in $K$-theory to relate information about the structure of tensor categories and their fusion rule algebras.

An invertible object $a$ is a simple object that can be characterized in several ways – either $d_a = 1$, or $a \otimes \bar{a} \simeq 1$, or $a \otimes x$ is simple (denote it $(ax)$ for every simple $x$. The invertible objects $I \subset S$ form a basis of a fusion subalgebra $A \subset R$ that is the group algebra of a finite abelian group. Let us lable by $a, b, c, \ldots$ invertible objects while leaving the letters $p, q, \ldots, x, y, \ldots$ for generic simple objects. The fusion ring $R$ breaks up into orbits, or simple modules, under the left/right action of $A$. This is the simplest situation of a fusion ring and a subring that we would like to analyse in particular its relevance to categorification.

In Conformal Field Theory language invertible objects correspond to simple currents. Simple currents have many application – in particular the analysis of the so called simple current extensions and construction of modular invariants [4,13,21].

## 3 Categorification

Categorification, *i.e.*, reversing the $K_0$ functor, is a very challenging problem. Some very initial "experimental" work of solving the pentagon equations to obtain categories from given fusion rules was done in [11]. For the fusion rules of truncated $sl(n)$ with the relevant Hecke algebra the corresponding braided tensor categories were reconstructed in [16]. The pentagon is a (in general a nonabelian) 3-cocycle condition – a preliminary sketch of how to attack the relevant nonabelian cohomology problem is given in [5]. For the case of abelian fusion rules ($R = A$ is the group algebra of an abelian group) it is an ordinary group cohomology problem solved in [9,18]. The categorification of the fusion rules of the quaternionic or the rank 8 dihedral group and their generalizations (where all but one of the sectors are abelian) was done in [22]. One would like to characterize the image of $K_0$ in the category of all fusion rule algebras and find "moduli" distinguishing categories with the same fusion rules. In a very recent paper [8] the authors have proved the vanishing of the Yetter cohomology implying Ocneanu's rigidity, *i.e.*, for each fusion ring there are a finite number of nonequivalent tensor categories mapped by $K_0$ to this ring. In the case of modular categories I have been tempted since a long time to conjecture [14] that the balancing phases (the $T$-matrix) separates (classes of equivalent) categories with the same fusion rules (="character table"=modular $S$-matrix), *i.e.*, modular fusion algebras provide a complete invariant for modular tensor categories.

## 4 Invertible objects

Suppressing the summation over the multiplicities the pentagon equation reads

$$F_{uy}^{pqv,t}\, F_{xv}^{urs,t} = \sum_w F_{uw}^{pqr,x} F_{xy}^{pws,t} F_{wv}^{qrs,y} \tag{1}$$

where the $6j$-symbols (tetrahedra) are defined by

$$((p \otimes q)_u \otimes r)_t \simeq \oplus_v\, F_{uv}^{pqr,t}\, (p \otimes (q \otimes r)_v)_t$$

with $(p \otimes q)_r$ denoting (again suppressing the multiplicities $N_{pq}^r$) the projection on a particular sector $r$ of the rhs of $p \otimes q = \cdots \oplus r \oplus \ldots$

Consider now particular cases of tetrahedra where some lable is from the set of invertible objects and because these are fixed by only three labels we denote

$$f_{abc} = F_{(ab)(bc)}^{abc,(abc)} \in T_6,$$

$$f_{abx} = F_{(ab)(bx)}^{abx,(abx)} \in T_3,$$

$$f_{axb} = F_{(ax)(xb)}^{axb,(axb)} \in T_2,$$

where we write $T_n$ for the set of tetrahedra with $n$ of the 6 labels corresponding to invertible objects. (We assume that the $S_4$ symmetry of $6j$-symbols [12] has been used.) The tetrahedra with one invertible object

$$f_{xay}^z = F_{(xa)(ay)}^{xay,z} \in T_1, \quad z \in \mathrm{supp}(xay),$$

$$f_{axy}^z = F_{(ax)(\bar{a}z)}^{axy,z} \in T_1, \quad z \in \mathrm{supp}(axy)$$

are square matrices of size $N^z_{xa,y} = N^z_{x,ay}$. For the above tetrahedra the pentagon equations become

$$P_{6^5}: \quad f_{(ab)cd}\, f_{ab(cd)} = f_{abc}\, f_{a(bc)d}\, f_{bcd} \tag{2}$$

$$P_{6,3^4}: \quad f_{(xa)bc}\, f_{xa(bc)} = f_{xab}\, f_{x(ab)c}\, f_{abc} \tag{3}$$

$$P_{3^2,2^3}: \quad f_{(ax)bc}\, f_{ax(bc)} = f_{axb}\, f_{a(xb)c}\, f_{xbc} \tag{4}$$

$$P_{3^2,1^3}: \quad f^z_{(xa)by}\, f^z_{xa(by)} = f_{xab}\, f^z_{x(ab)y}\, f_{aby}, \tag{5}$$

$$P_{2,1^4}: \quad f^z_{(ax)by}\, f^z_{ax(by)} = f_{axb}\, f^z_{a(xb)y}\, f^z_{xby}, \tag{6}$$

where in the last two lines we again have suppressed summation over multiplicities and where $z \in \mathrm{supp}(xaby)$ and $z \in \mathrm{supp}(axby)$ respectively. One sees that (2) is exactly the 3-cocycle condition of abelian group cohomology written multiplicatively and analyzed in details in [7] and [9]. Thus the multiplication between invertible objects is associative up to this cocycle. Considering the action of the invertible objects on generic ones we have the module property up to a 3-cocycle in $T_3$ satisfying the conditions $P_{6,3^4}$ and the bimodule property up to a 3-cocycle in $T_2$ satisfying $P_{3^2,2^3}$. The first two are described as cocycle conditions in [7] while a straightforward generalization of the bar complex used there will describe also the third set of pentagons. (Note that in all these pentagons there are no summations so 'taking logarithms' we arrive at usual cocycle conditions.) By appropriate gauge choice one can simplify the last two sets of equations and turn them again into usual cohomology conditions. (This will be explained in details elsewhere.) They fix the tetrahedra from $T_1$ which are the 'obstruction' to the relative tensor product of $A$ modules. A simple generalization of [22] shows that if $B \subset A$ is the subalgebra of elements that are fixed by some element of $S$ then the tetrahedra from $T_6$ or $T_3$ with a lable from $B$ trivialize while the ones from $T_2$ with a lable from $B$ provide a bicharacter of $B$.

Once the tetrahedra from $T_1$ have been determined we have an 'action' of the invertible objects on a generic $6j$-symbol

$$f^t_{ua(\bar{a}v)}\, F^{pqv,t}_{uy} = f^{(ua)}_{pqa}\, f^y_{qa(\bar{a}v)}\, F^{p(qa)(\bar{a}v),t}_{(ua)y} \tag{7}$$

(again summation over multiplicities has been suppressed while there is no summation over labels). *I.e.*, if we want to solve the pentagon for a fusion ring that has an abelian fusion subring the tetrahedra from $T_n$ can be determined by ordinary abelian group cohomology while for the generic tetrahedra one effectively has to consider them as different unknowns only if their labels are from different orbits of the abelian fusion ring. This together with $S_4$ symmetry will considerably reduce the number of unknowns and make the pentagons manageable even for relatively larger fusion rings.

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